Electromagnetic Casimir energy with extra dimensions

H. Alnes, F. Ravndal and I.K. Wehus

Department of Physics, University of Oslo, N-0316 Oslo, Norway.

and

K. Olaussen

Department of Physics, NTNU, N-7491 Trondheim, Norway.

Abstract

We calculate the energy-momentum tensor due to electromagnetic vacuum fluctuations between two parallel hyperplanes in more than four dimensions, considering both metallic and MIT boundary conditions. Using the axial gauge, the problem can be mapped upon the corresponding problem with a massless, scalar field satisfying respectively Dirichlet or Neumann boundary conditions. The pressure between the plates is constant while the energy density is found to diverge at the boundaries when there are extra dimensions. This can be related to the fact that Maxwell theory is then no longer conformally invariant. A similar behavior is known for the scalar field where a constant energy density consistent with the pressure can be obtained by improving the energy-momentum tensor with the Huggins term. This is not possible for the Maxwell field. However, the change in the energy-momentum tensor with distance between boundaries is finite in all cases.

1 Introduction

When a classical field is quantized, the modes that can be excited are solutions of the classical wave equation and are labeled by different quantum numbers. These modes will depend on the imposed boundary conditions and will therefore be influenced by the presence of confining boundaries. Each such mode has a zero-point energy which contributes to the total vacuum energy of the field. As a result, these vacuum fluctuations give the ground state of the system an energy which depends on the presence of nearby boundaries. For two parallel and perfectly conducting plates placed in vacuum Casimir[1] showed that for the electromagnetic field this energy corresponds to an attractive pressure

$$P = -\frac{\pi^2}{240L^4} \tag{1}$$

between the plates, separated by a distance L. This macroscopic quantum effect was for a long time in doubt, even after the first experimental verifications by Sparnaay[2]. Today this Casimir force is measured to high precision[3] and even effects of non-zero temperatures are being investigated[4].

1

Together with this experimental progress, modern regularization methods to remove the unphysical divergences endemic in these calculations now make them much simpler than previously[5]. One can then investigate more detailed properties of the effect like how the energy or stresses are distributed between the plates. One must then calculate the vacuum expectation value of the full energy-momentum tensor as was done by Lütken and Ravndal[6]. This confirmed a previous calculation by de Witt[7] of the fluctuations of the electric and magnetic fields near a metallic boundary where they diverge but in such a way that the energy density remains finite. Since Maxwell theory is conformally invariant in $D = 4$ spacetime dimensions, one can directly relate this energy density to the attractive pressure[8].

Recently the Casimir energy has been invoked to explain the dark energy which seems to drive the present acceleration of the Universe[9]. It appears in particular in cosmological models with extra dimensions[10]. In these models the confinement of the fluctuating fields is provided by compactification of the extra dimensions. The energy appears as a cosmological constant in our four-dimensional Universe and is given numerically by a formula of the same form as (1) with L given by the size of the compactified dimensions.

The calculation of Casimir energies in higher-dimensional spacetimes was first done by Ambjørn and Wolfram[11]. They calculated the global energies equivalent to the Casimir force and thus obtained no knowledge of how the energy is distributed between the hyperplanes. Since the force due to electromagnetic fluctuations is expected to be proportional to the force due to fluctuations of a massless scalar field, only the effects of this kinematically simpler field was investigated. Each mode has a momentum $\mathbf{k}_T$ transverse to the normal of the plates plus a component $k_z = n\pi/L$ with $n = 1, 2, \ldots$ in the direction of the normal. If d is the number of spatial dimensions, the vacuum energy between the plates per $(d-1)$-dimensional hyperarea will follow from the divergent integral

$$E = \frac{1}{2} \sum_{n=1}^{\infty} \int \frac{\mathrm{d}^{d-1} k_T}{(2\pi)^{d-1}} \sqrt{\mathbf{k}_T^2 + (n\pi/L)^2} \tag{2}$$

We can now do the transverse integration by dimensional regularization. The remaining sum over n is then done by analytical continuation of the Riemann zeta function $\zeta_R(z)$ to give

$$E = -\frac{\Gamma(-d/2)\zeta_R(-d)}{2(4\pi)^{d/2}} \left(\frac{\pi}{L}\right)^d \tag{3}$$

It can be simplified using the reflection formula

$$\Gamma(s/2)\pi^{-s/2}\zeta_R(s) = \Gamma((1-s)/2)\pi^{-(1-s)/2}\zeta_R(1-s) \tag{4}$$

for the zeta function. Then we can write $E = \mathcal{E}_0 L$ when we introduce the energy density

$$\mathcal{E}_0 = -\frac{\Gamma(D/2)\zeta_R(D)}{(4\pi)^{D/2} L^D} \tag{5}$$

2

Defining now the pressure between the plates by $P = -\partial E/\partial L$, it is then simply

$$P = (D - 1)\mathcal{E}_0 \qquad (6)$$

where $D = 1 + d$ is the spacetime dimension. Taking $D = 4$ and multiplying the result by two for the two polarization degrees of the photon, we recover the original result (1). In D-dimensional spacetime we must for the same reason multiply the result (6) by $D - 2$ to obtain the electromagnetic Casimir force.

The spatial distribution of the vacuum energy can be obtained from the energy-momentum tensor. For the scalar field it is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L} \qquad (7)$$

where the massless Lagrangian is $\mathcal{L} = (1/2)(\partial_\lambda\phi)^2$ choosing the metric to be $\eta_{\mu\nu} = \text{diag}(1, -1, \cdots, -1)$. Since its trace $T^\mu_\mu = (1 - D/2)(\partial_\lambda\phi)^2$ is zero only in $D = 2$ dimensions, it is in general not conformally invariant in higher dimensions. Calculating now the Casimir energy density $\langle T_{00}\rangle$ in for example $D = 4$ spacetime dimensions, one then obtains a result which diverges at the plates after regularization[12]. When integrated, it will thus not reproduce the total Casimir energy corresponding to the force (6).

For the scalar field this apparent problem can be solved. One can improve[13] the above energy-momentum tensor in any spacetime dimension $D > 2$ by adding the Huggins term[14]

$$\Delta T_{\mu\nu} = -\frac{1}{4}\frac{D - 2}{D - 1}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2 \qquad (8)$$

The energy-momentum tensor is then traceless and conformal invariance has been restored. Since this new term is a divergence, it will not contribute to the force, but change the distribution of energy around the plates. Vacuum expectation values of all components of the energy-momentum tensor are now constant between the plates and zero outside as already noticed by de Witt[7], Milton[15] and others[12].

However, the situation for the electromagnetic field is somewhat different. It has the energy-momentum tensor

$$T_{\mu\nu} = F_{\mu\alpha}F^\alpha_{\ \nu} + \frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \qquad (9)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor. From the trace $T^\mu_\mu = (-1 + D/4)F^2_{\alpha\beta}$ we see that it is conformally invariant only in $D = 4$ spacetime dimensions. In this case the Casimir energy is also constant between the plates[6] as for the scalar field. But for dimensions $D > 4$ it is no longer clear how the energy is distributed since there is no way to construct a gauge-invariant analogue of the Huggins term in Maxwell theory.

In the following we will investigate this problem in more detail. In $D = 4$ dimensions one can choose the transverse gauge and expand the classical field in electromagnetic multipoles. This is cumbersome in higher dimensions since the field then has more magnetic

than electric components. From the geometry of the problem it is more natural to choose the axial gauge $n^\mu A_\mu = A_z = 0$ where the D-vector $n^\mu = (0, 0, \ldots, 0, 1)$ is normal to the plates and is called the z-direction. The Faraday tensor then has a correlator which can be directly obtained from scalar field theory in the same geometry. In the next chapter we therefore derive a general expression for the scalar field correlators, both in the case of Neumann and Dirichlet boundary conditions. These can then be used to calculate the expectation value of the scalar energy-momentum tensor (7) and the relevance of the Huggins term is discussed.

For the electromagnetic field considered in Chapter 3, we need to know the boundary conditions. These can be of the metallic type used for the standard Casimir force in $D = 4$ or the QCD version used in the MIT bag model for confinement of quarks. In the axial gauge we find that these two possibilties correspond to Dirichlet and Neumann conditions for the corresponding scalar field. The fluctuations of the different components of the Faraday tensor are then calculated with particular attention to the energy density and the pressure between the plates. While the pressure is found to be constant and in agreement with the global Casimir force, the energy density diverges at the plates.

In the last chapter this problem, which no longer can be cured with a Huggins term, is discussed and compared with similar divergences in other systems. With physical boundaries that only confines fluctuations with frequencies below a certain cut-off, all field fluctuations should reach a finite value when the boundaries are approached.

2   Scalar fields

It will be very convenient to denote by a bar any vector or tensor component orthogonal to the unit normal n^μ of the plates, which is taken to be in the z-direction. Thus for a full D-vector we write $A = (A^\mu) = (A^{\bar\mu}, A^z)$. Note that the $(D-1)$-vector $\bar A = (A^{\bar\mu})$ also includes the time component A^0. The metric can thus be written as $\eta^{\mu\nu} = \bar\eta^{\mu\nu} - n^\mu n^\nu$ where $\bar\eta$ is the projection of η onto the barred subspace. The field operator for a massless scalar field satisfying the Dirichlet boundary conditions $\phi(\bar x, z = 0, L) = 0$ in $D = d + 1$ spacetime dimensions will then be

$$\phi(x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \int \frac{\mathrm{d}^{d-1} k_T}{(2\pi)^{d-1}} \sqrt{\frac{1}{2\omega_n}} \left[a_n(\mathbf{k}_T) \, e^{-i\bar k \cdot \bar x} + a_n^\dagger(\mathbf{k}_T) \, e^{i\bar k \cdot \bar x} \right] \sin(n\pi z/L) \qquad (10)$$

Here $\bar k = (\omega_n, \mathbf{k}_T)$ with the frequency $\omega_n = \sqrt{\mathbf{k}_T^2 + m_n^2}$ where $m_n = k_z = n\pi/L$. Had we instead chosen the Neumann boundary condition $\partial_z \phi(\bar x, z = 0, L) = 0$, we just have to make the replacement $\sin(n\pi z/L) \to \cos(n\pi z/L)$ in the sum. Then there should also be a $n = 0$ mode to be included in the sum with normalization constant $\sqrt{1/L}$. But with the regularization we will use in the following, it will not contribute and is therefore not further considered.

2.1 Feynman correlator

In the above field operator for the scalar Dirichlet modes we have used a normalization which corresponds to $[a_n(\mathbf{k}_T), a_{n'}^\dagger(\mathbf{k}'_T)] = \delta_{nn'}(2\pi)^{d-1}\delta(\mathbf{k}_T - \mathbf{k}'_T)$ for the annihilation and creation operators. From this we find the Feynman propagator

$$
\begin{aligned}
G_D(x, x') &= \langle \Omega_D | T\phi(x)\phi(x') | \Omega_D \rangle \\
&= i \int \frac{d^d \bar{k}}{(2\pi)^d} \frac{2}{L} \sum_{n=1}^{\infty} \frac{\sin(n\pi z/L)\sin(n\pi z'/L)}{\bar{k}^2 - m_n^2 + i\varepsilon} \, e^{-i\bar{k}\cdot(\bar{x}-\bar{x}')}
\end{aligned}
\tag{11}
$$

Here we integrate over all components of the d-dimensional Lorentz vector $\bar{k}$. Assuming $\bar{\epsilon} \equiv \bar{x} - \bar{x}'$ to be spacelike, we may choose a coordinate system where it has no components in the time direction. We can then rotate k^0 to the imaginary axis and find

$$
G_D(x, x') = \frac{2}{L} \int \frac{d^d \bar{k}}{(2\pi)^d} \sum_{n=1}^{\infty} \frac{\sin(n\pi z/L)\sin(n\pi z'/L)}{\bar{k}^2 + m_n^2} \, e^{i\bar{k}\cdot\bar{\epsilon}}
\tag{12}
$$

(where now $\bar{k}$ is a Euclidean vector. In order to evaluate the sum, we consider the function

$$
g(z, z') = \frac{2}{L} \sum_{n=1}^{\infty} \frac{\sin(n\pi z/L)\sin(n\pi z'/L)}{\bar{k}^2 + m_n^2}
\tag{13}
$$

which solves the differential equation

$$
\left(-\frac{d^2}{dz^2} + \bar{k}^2 \right) g(z, z') = \delta(z - z')
\tag{14}
$$

on $[0, L]$ with Dirichlet boundary conditions. As can be verified by insertion, the solution is

$$
g(z, z') = \frac{\sinh \bar{k} z_< \sinh \bar{k}(L - z_>)}{\bar{k} \sinh \bar{k} L}
\tag{15}
$$

where $z_< = \min(z, z')$ and $z_> = \max(z, z')$. Expanding the hyperbolic functions, one then finds

$$
\begin{aligned}
g(z, z') &= \frac{1}{2\bar{k}} \left(e^{-\bar{k}|z-z'|} - e^{-\bar{k}(z+z')} - e^{-\bar{k}(2L-z-z')} + e^{-\bar{k}(2L-|z-z'|)} \right) \sum_{j=0}^{\infty} e^{-2j\bar{k}L} \\
&= \sum_{j=-\infty}^{\infty} \int \frac{dk_z}{2\pi} \frac{1}{\bar{k}^2 + k_z^2} \left[e^{ik_z(z-z'-2jL)} - e^{ik_z(z+z'-2jL)} \right].
\end{aligned}
\tag{16}
$$

The last equality is verified by evaluating the k_z integral by contour integration. Inserting now this partial result into (12) and using rotational invariance, we find

$$
G_D(x, x') = \sum_{j=-\infty}^{\infty} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{k^2} \left(e^{i[k_z(z-z'-2jL)+\bar{k}\cdot\bar{\epsilon}]} - e^{i[k_z(z+z'-2jL)+\bar{k}\cdot\bar{\epsilon}]} \right)
\tag{17}
$$

5

where now the $(d+1)$-dimensional vector $k = (\bar{k}, k_z)$. Each of the integrals are given by the generalized Coulomb potential

$$V_n(\mathbf{x} - \mathbf{x}') = \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{1}{k^2} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} = \frac{\Gamma(n/2 - 1)}{4\pi^{n/2} |\mathbf{x} - \mathbf{x}'|^{n-2}} \tag{18}$$

in $n = d+1$ spatial dimensions. Had we instead considered Neumann boundary conditions, the sine function in (10) would have been replaced by the corresponding cosine function. The only change would then have been that the last term in (17) came in with opposite sign. Introducing the D-vectors $z_j = (\bar{\epsilon}, z - z' - 2jL)$ and $\tilde{z}_j = (\bar{\epsilon}, z + z' - 2jL)$ of lengths $R_j = (z_j^2)^{1/2}$ and $\tilde{R}_j = (\tilde{z}_j^2)^{1/2}$, we can now write the result for both correlators as

$$G(x, x')_{N/D} = \sum_{j=-\infty}^{\infty} \left[V_{d+1}(R_j) \pm V_{d+1}(\tilde{R}_j) \right] \tag{19}$$

where the upper sign is for Neumann and the lower for Dirichlet boundary conditions. For a massive field we would have found a similar result, but with the Coulomb potential replaced by the corresponding generalized Yukawa potential.

In fact, almost every student of introductory electrostatics could have written down this result immediately by realizing that the problem is equivalent to calculating the potential of a point charge between parallel plates in $D = d+1$ spatial dimensions, using the method of images to enforce the boundary conditions.

2.2 Energy-momentum tensor

The term $V_{d+1}(R_0)$ in (19) is equal to the free correlator $G_0(x, x') = \langle 0 | T\phi(x)\phi(x') | 0 \rangle$, where $|0\rangle$ is the bulk vacuum. It diverges in the limit $x' \to x$. But defining now the physical vacuum expectation value of the energy-momentum tensor by the point-split limit

$$\langle T_{\mu\nu}(x) \rangle = \lim_{x' \to x} [\langle \Omega | T_{\mu\nu}(x', x) | \Omega \rangle - \langle 0 | T_{\mu\nu}(x', x) | 0 \rangle] \tag{20}$$

its contribution is removed. The finite expectation values will then follow from the regularized correlator $G_D(x, x') - G_0(x, x')$ which contains the effects of the plates. We will continue to denote it by $G_D(x, x')$ in the following and it is given by (19) when we in the first part leave out the $j = 0$ term. Since we have assumed that $x - x'$ is non-zero and spacelike, the time-ordering symbol in the correlator can be ignored and it satisfies the Klein-Gordon equation $(\bar{\partial}^2 - \partial_z^2 + m^2)G(x, x') = 0$ for both boundary conditions.

For the Dirichlet vacuum expectation value of the scalar energy momentum tensor (7), we first need the part

$$\langle \partial_x^{\bar{\mu}} \phi(x) \partial_{x'}^{\bar{\nu}} \phi(x') \rangle_D \;\; = \;\; -\partial_x^{\bar{\mu}} \partial_{x'}^{\bar{\nu}} G_D(x, x') = -\frac{1}{d} \eta^{\bar{\mu}\bar{\nu}} \bar{\partial}^2 G_D(x, x')$$

$$= \;\; \frac{1}{d} \eta^{\bar{\mu}\bar{\nu}} (m^2 - \partial_z^2) G_D(x, x') \tag{21}$$

6

using Lorentz invariance. Similarly, it follows that

$$\langle \partial_z \phi(x) \partial_{z'} \phi(x') \rangle_D = -\partial_z^2 G_N(x, x') \tag{22}$$

since the two parts in the correlator (19) has opposite symmetry under the exchange $z \to z'$. For vacuum expectation value of the point-split Lagrangian

$$\mathcal{L}(x, x') = \frac{1}{2}[\eta_{\bar\mu\bar\nu}\partial_x^{\bar\mu}\phi(x)\partial_{x'}^{\bar\nu}\phi(x') - \partial_z\phi(x)\partial_{z'}\phi(x') - m^2\phi(x)\phi(x')] \tag{23}$$

we thus find

$$\langle \mathcal{L}(x, x') \rangle_D = \frac{1}{2}\partial_z^2\Big[G_N(x, x') - G_D(x, x')\Big] \tag{24}$$

The point-split expressions for the canonical energy-momentum tensor (7) are thus found to be

$$\langle T_{\bar\mu\bar\nu} \rangle_D = \eta_{\bar\mu\bar\nu}\Big(\frac{m^2}{d}G_D - \partial_z^2\Big[\frac{1}{d}G_D + \frac{1}{2}(G_N - G_D)\Big]\Big) \tag{25}$$

$$\langle T_{zz} \rangle_D = -\frac{1}{2}\partial_z^2(G_N + G_D) \tag{26}$$

Corresponding results for the Neumann expectation values are obtained by the exchange $D \leftrightarrow N$. The physical limit $x \to x'$ can now be taken where a resulting z-dependence can only come from the last sum in the correlators (19). But for the pressure $P = \langle T_{zz} \rangle$ we see that this will cancel out in the sum $G_D + G_N$ so that the pressure is constant between the plates. This is physical reasonable and is also the case for the fluctuations of a massive field. It follows directly from the conservation of the energy-momentum tensor. The expectation values of the other other components of the energy-momentum tensor in (25) will in general be dependent on the position z between the plates.

Let us now calculate the pressure in the massless limit. We will then need the double derivative $\partial_z^2(G_D + G_N)$ which follows directly from (19) in the limit $x \to x'$ as

$$\partial_z^2(G_N + G_D) = 2\lim_{z' \to z}\sum_{j=-\infty}^{\infty}{}' \partial_z^2 V_{d+1}(R_j) = 2d(d-1)\frac{\Gamma((d-1)/2)}{4\pi^{(d+1)/2}}\sum_{j=-\infty}^{\infty}{}'\frac{1}{|2jL|^{d+1}}$$

$$= \frac{2(D-1)\Gamma(D/2)}{(4\pi)^{D/2}L^D}\zeta_R(D) \tag{27}$$

where the $'$ denotes that $j = 0$ is excluded from sum. Using this in (26) we reproduce exactly the standard pressure (6) obtained from the total energy. It is seen to be the same for both boundary conditions.

The energy density $\mathcal{E} = \langle T_{00} \rangle$ between the plates follows from (25). When the mass $m = 0$, we then need to calculate in addition the quantity

$$\partial_z^2(G_N - G_D) = 2\lim_{z' \to z}\sum_{j=-\infty}^{\infty}\partial_z^2 V_{d+1}(\tilde R_j)$$

$$= \frac{(D-1)\Gamma(D/2)}{(4\pi)^{D/2}L^D}f_D(z/L) \tag{28}$$

7

when we introduce the function

$$f_D(z/L) = \sum_{j=-\infty}^{\infty} \frac{1}{|j + z/L|^D} \tag{29}$$

Notice that the term $j = 0$ is now to be included. The sum can be expressed by the Hurwitz zeta function

$$\zeta_H(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} \tag{30}$$

which allows us to write

$$
\begin{aligned}
f_D(z/L) &= \sum_{j=0}^{\infty} \frac{1}{(j + z/L)^D} + \sum_{j=1}^{\infty} \frac{1}{(j - z/L)^D} \\
&= \zeta_H(D, z/L) + \zeta_H(D, 1 - z/L)
\end{aligned}
\tag{31}
$$

When $D = 4$ the same position-dependent term was derived on this form by Kimball[15]. But when the spacetime dimension D is an even number, we can express the result in terms of the digamma function $\psi(x)$ using the relation

$$\zeta_H(k, x) = \frac{(-1)^k}{(k-1)!} \left(\frac{d}{dx}\right)^{k-1} \psi(x) \tag{32}$$

We then have

$$f_D(z/L) = \frac{1}{(D-1)!} \left(\frac{d}{dx}\right)^{D-1} \left[\psi(x) - \psi(1-x)\right] \qquad (D = \text{even}) \tag{33}$$

where $x = z/L$. This simplifies even more since $\psi(x) - \psi(1-x) = -\pi \cot(\pi x)$, which allows us to write

$$f_D(z/L) = \frac{\pi^D}{\Gamma(D)} \left(-\frac{d}{d\theta}\right)^{D-1} \cot\theta \qquad (D = \text{even}) \tag{34}$$

with $\theta = \pi z/L$. For the ordinary Casimir effect in $D = 4$ spacetime dimensions, this function also appeared in the calculation of the electromagnetic field using another regularization and choice of gauge[6]. This follows from writing $f_4(z/L) = (\pi^4/3)F(\theta)$ which gives

$$F(\theta) = \left(\frac{3}{\sin^4\theta} - \frac{2}{\sin^2\theta}\right) \tag{35}$$

This function will then characterize all position-dependent expectation values when $D = 4$.

Collecting the above results, we now have the scalar vacuum energy density in arbitrary spacetime dimensions

$$\mathcal{E}_{D/N} = -\frac{\Gamma(D/2)}{(4\pi)^{D/2} L^D} \left[\zeta_R(D) \pm (D/2 - 1)f_D(z/L)\right] \tag{36}$$

8

where the lower sign is for Neumann boundary conditions. While the first term corresponds to a constant density, the last term gives a position-dependent contribution which in general diverges at the position of the plates, i.e. where $z = 0$ and $z = L$. Only in the special case $D = 2$ when the scalar field has conformal invariance, will it be absent. When integrated over the volume between the plates, the first term alone is seen to give the total Casimir energy(5). The last term gives a divergent contribution to the same energy and should be absent. No such term was found in the calculation of the pressure. It is consistent with just the first part of the energy density which alone gives the correct Casimir force.

2.3 Huggins term

A free, massless scalar field can couple to gravity in a conformally invariant way. The resulting energy-momentum tensor will thus be traceless[7][13]. It differs from the canonical expression (7) by the extra Huggins term (8). Using the equation of motion $\partial^2 \phi = 0$, one then finds that the improved energy-momentum tensor indeed is traceless.

When we now want to evaluate the Huggins term for the vacuum between the two plates using the above point-split regularization, we interpret

$$\partial_\mu \partial_\nu \phi^2 = (\partial_\mu \partial_\nu + \partial_\mu \partial_\nu' + \partial_\mu' \partial_\nu + \partial_\mu' \partial_\nu') \phi(x) \phi(x')$$

where the primed derivatives are with respect to x'. This gives the ground state expectation value

$$\langle \Delta T_{\mu\nu} \rangle_D = \frac{1}{2} \bar{\eta}_{\mu\nu} \frac{D - 2}{D - 1} \partial_z^2 (G_N - G_D) \tag{37}$$

which is seen to be proportional to $\langle \mathcal{L} \rangle$. The Huggins correction has no components in the z-direction, leaving the pressure unaltered. The z-dependent terms cancel against the same terms in the canonical part (25) so that the resulting energy density will be constant. In fact, when $m = 0$ we have for both sets of boundary conditions that

$$\langle T_{\mu\nu} + \Delta T_{\mu\nu} \rangle = \mathcal{E}_0 (\bar{\eta}_{\mu\nu} + (D - 1) n_\mu n_\nu) \tag{38}$$

when expressed in terms of the energy density (5). The pressure is simply $D - 1$ times this constant energy density. This is a direct consequence of the energy-momentum tensor now being traceless.

3 Maxwell fields

The electromagnetic Lagrangian density $\mathcal{L} = -F_{\mu\nu}^2/4$ is gauge invariant. In a general spacetime it can be written as

$$\mathcal{L} = \frac{1}{2} E_i^2 - \frac{1}{4} B_{ij}^2 \tag{39}$$

9

where $E_i = -(\dot{A}_i + \partial_i A^0)$ are the components of the electric field vector while the magnetic field is given by the antisymmetric tensor $B_{ij} = \partial_i A_j - \partial_j A_i$.

We want to calculate the vacuum expectation value of the corresponding energy-momentum tensor (9) between the plates. As already explained in the introduction, it is then most natural to work in the axial gauge $n^\mu A_\mu = 0$. Since the plate normal vector n^μ only has a component along the z-axis, this requires the component $A_z = 0$. The component A_0 is no longer a canonical variable, but depends on the others via the Maxwell equation $\partial_i F^{i0} = 0$, which gives

$$A_0 = -\Delta^{-1}\partial_i \dot{A}_i \tag{40}$$

where the operator $\Delta = \partial_i^2$. There are thus $D - 2$ independent degrees of freedom in a D-dimensional spacetime described by the spatial field components A_i where $i \neq z$. We can then express the full Lagrangian in terms of these fields. After partial integrating and neglecting surface terms, we find it to be

$$L = \frac{1}{2}\int \mathrm{d}^d x \left[\dot{A}_i\left(\delta_{ij} - \partial_i \Delta^{-1}\partial_j\right)\dot{A}_j - A_i\left(\partial_i \partial_j - \delta_{ij}\Delta\right)A_j\right] \tag{41}$$

As usual, the first or electric part acts like a kinetic energy while the magnetic part acts like a potential energy.

3.1 Boundary conditions and the correlator

In order to quantize this theory, we must impose boundary conditions for the electromagnetic field components on the confining plates. For the original Casimir effect in $d = 3$ dimensions one had metallic plates in mind where one could impose the standard constraints $\mathbf{n} \times \mathbf{E} = 0$ and $\mathbf{n} \cdot \mathbf{B} = 0$ on the elecric and magnetic field vectors. For the more abstract case we have in mind here, we could just as well consider the MIT boundary condition $n^\mu F_{\mu\nu} = 0$ proposed at MIT for the quark bag model ensuring color confinement[16]. In terms of components, this is equivalent to the two conditions $\mathbf{n} \times \mathbf{B} = 0$ and $\mathbf{n} \cdot \mathbf{E} = 0$. They are therefore just the electromagnetic duals of the metallic boundary conditions.

For our problem under consideration the MIT boundary condition can most directly be imposed. With the normal vector n^μ along the z-axis, it is equivalent to $F_{\bar{\mu}z} = 0$ in our previous index notation. Now in the axial gauge this is simply equivalent to the Neumann boundary condition $\partial_z A_i(\bar{x}, z = 0, L) = 0$, and defining Δ^{-1} in (40) with Neumann boundary conditions.

With this index notation, the metallic boundary conditions can also easily be generalized to higher dimensions by noticing that in $d = 3$ dimensions they correspond to $F_{\bar{\mu}0} = 0$. In the axial gauge this is achieved in all dimensions by requiring $A_i(\bar{x}, z = 0, L) = 0$, i.e. Dirichlet conditions, and defining Δ^{-1} in (40) with Dirichlet boundary conditions. In this way we can take directly over many of the previous results for the scalar field.

10

The field components $A_{\bar{\mu}}$ obey the classical wave equation $\partial^2 A_{\bar{\mu}} - \partial_{\bar{\mu}}(\partial^{\bar{\nu}} A_{\bar{\nu}}) = 0$. Solutions will be of the form $A_{\bar{\mu}}(x) = a_{\bar{\mu}}(\bar{x})b(z)$ where the function $b(z) \propto \cos(n\pi z/L)$ when we impose MIT, i.e. Neumann, boundary conditions and $b(z) \propto \sin(n\pi z/L)$ for metallic or Dirichlet boundary conditions. With these boundary conditions the remaining functions $a_{\bar{\mu}}(\bar{x})$ satisfy $(\bar{\partial}^2 + m_n^2)a_{\bar{\mu}} - \partial_{\bar{\mu}}(\partial^{\bar{\nu}} a_{\bar{\nu}}) = 0$ which are just the equations of motion for a massive vector field with mass $m_n = n\pi/L$. With this observation, we then immediately have the correlator

$$
\begin{aligned}
D_{\bar{\mu}\bar{\nu}}(x, x') &= \langle \Omega_D | T A_{\bar{\mu}}(x) A_{\bar{\nu}}(x') | \Omega_D \rangle \\
&= i \int \frac{d^d \bar{k}}{(2\pi)^d} \frac{2}{L} \sum_{n=1}^{\infty} \frac{\sin(n\pi z/L)\sin(n\pi z'/L)}{\bar{k}^2 - m_n^2 + i\varepsilon} \left(\eta_{\bar{\mu}\bar{\nu}} - \frac{k_{\bar{\mu}}k_{\bar{\nu}}}{m_n^2} \right) e^{-i\bar{k}\cdot(\bar{x}-\bar{x}')} \quad (42)
\end{aligned}
$$

A corresponding result is obtained with Neumann boundary conditions. As before, we then drop the mode with $n = 0$.

3.2 Electromagnetic expectation values

We are now in the position of calculating the value of the energy-momentum tensor (9) between the two plates. First we need the expectation value

$$
\begin{aligned}
\langle F_{\bar{\mu}\bar{\nu}}(x) F_{\bar{\alpha}\bar{\beta}}(x') \rangle_D &= \frac{2}{d}\left(\eta_{\bar{\alpha}\bar{\mu}}\eta_{\bar{\beta}\bar{\nu}} - \eta_{\bar{\alpha}\bar{\nu}}\eta_{\bar{\beta}\bar{\mu}} \right)\bar{\partial}^2 G_D(x, x') \\
&= \frac{2}{D-1}\left(\eta_{\bar{\alpha}\bar{\mu}}\eta_{\bar{\beta}\bar{\nu}} - \eta_{\bar{\alpha}\bar{\nu}}\eta_{\bar{\beta}\bar{\mu}} \right)\partial_z^2 G_D(x, x') \quad (43)
\end{aligned}
$$

since the massless correlator satisfies the free wave equation $(\bar{\partial}^2 - \partial_z^2)G_D = 0$. The structure of this result follows directly from antisymmetry of the field tensor and Lorentz invariance in the barred directions. As expected, it is simply given by the scalar correlator.

In the same way we also find

$$
\langle F_{\bar{\mu}z}(x) F_{\bar{\nu}z}(x') \rangle_D = \eta_{\bar{\mu}\bar{\nu}}\left(\partial_z^2 - \frac{1}{d}\bar{\partial}^2 \right)G_D(x, x') = \frac{D-2}{D-1}\eta_{\bar{\mu}\bar{\nu}}\partial_z^2 G_N(x, x') \quad (44)
$$

The point-split expectation value of the Lagrangian density $\mathcal{L} = -F_{\bar{\mu}\bar{\nu}}^2/4 - F_{\bar{\mu}z}^2/2$ is therefore

$$
\langle \mathcal{L}(x, x') \rangle_D = \frac{1}{2}(D-2)\partial_z^2\left[G_N(x, x') - G_D(x, x') \right] \quad (45)
$$

in analogy with (24).

From these results we can now read off the fluctations of the vacuum fields when the limit $x \to x'$ is taken. For this purpose we combine (27) and (28) which give

$$
\partial_z^2 G_{N/D} = (D-1)\frac{\Gamma(D/2)}{(4\pi)^{D/2}L^D}\left[\zeta_R(D) \pm \frac{1}{2}f_D(z/L) \right] \quad (46)
$$

11

in this limit. With $\bar{\mu} = \bar{\nu} = 0$ in (44) we then find for the z-component of the electric field

$$\langle E_z^2 \rangle_D = (D-2)\frac{\Gamma(D/2)}{(4\pi)^{D/2}L^D}\left[\zeta_R(D) + \frac{1}{2}f_D(z/L)\right] \tag{47}$$

The other components follow from (43) which gives

$$\langle E_i^2 \rangle_D = -\frac{2\Gamma(D/2)}{(4\pi)^{D/2}L^D}\left[\zeta_R(D) - \frac{1}{2}f_D(z/L)\right] \tag{48}$$

where there is no summation over the index $i \neq z$ on the left-hand side. For the fluctuations in the magnetic components we similarly find $\langle B_{iz}^2 \rangle_D = -\langle E_z^2 \rangle_D$ and $\langle B_{ij}^2 \rangle_D = -\langle E_i^2 \rangle_D$ where again there is no summation over the indices $i, j \neq z$. These relations also hold for the case of Neumann boundary conditions except for a change of signs in the last term of (47) and (48).

These vacuum field fluctuations where first calculated by Lütken and Ravndal[6] for the ordinary Casimir effect with $D = 4$ spacetime dimensions in the Coloumb gauge and a different regularization. Using $\zeta_R(4) = \pi^4/90$ and the function (35) for $f_4(z/L)$ in the above general results, we find

$$\langle E_z^2 \rangle = \frac{\pi^2}{48L^4}\left[F(\theta) + \frac{1}{15}\right] \tag{49}$$

for Dirichlet boundary conditions, where $\theta = \pi z/L$ as before. The two other transverse components are given by (48) as

$$\langle E_x^2 \rangle = \langle E_y^2 \rangle = \frac{\pi^2}{48L^4}\left[F(\theta) - \frac{1}{15}\right] \tag{50}$$

Fluctuations of the transverse magnetic components are then $\langle B_x^2 \rangle = \langle B_y^2 \rangle = -\langle E_z^2 \rangle$ while for the normal compoent we have $\langle B_z^2 \rangle = -\langle E_x^2 \rangle$.

Since the pressure is given by the expectation value of $T_{zz} = F_{z\bar{\mu}}F^{\bar{\mu}}{}_z + \mathcal{L}$, it now follows as

$$P = -\frac{1}{2}(D-2)\partial_z^2(G_N + G_D) = (D-2)(D-1)\mathcal{E}_0 \tag{51}$$

when expressed in terms of the energy density (5). It is again constant between the plates and a factor $D-2$ times the scalar pressure (26). This is exactly as expected since the Maxwell field has $D-2$ scalar degrees of freedom.

For the other components of the energy-momentum tensor we similarly find

$$\begin{aligned}\langle T_{\bar{\mu}\bar{\nu}} \rangle_{D/N} &= -\frac{1}{2}\frac{D-2}{D-1}\partial_z^2\left[G_N + G_D \pm (D-4)(G_N - G_D)\right]\eta_{\bar{\mu}\bar{\nu}} \\ &= -(D-2)\frac{\Gamma(D/2)}{(4\pi)^{D/2}L^D}\left[\zeta_R(D) \pm (D/2-2)f_D(z/L)\right]\eta_{\bar{\mu}\bar{\nu}} \end{aligned} \tag{52}$$

12

where the lower sign is for Neumann boundary conditions. It is only for $D = 4$ that the last, position-dependent term will be absent. And it is also then that Maxwell theory is conformally invariant.

The above results for the electromagnetic field are very similar to what we found for the canonical, massless scalar field in the previous section. In that case the theory could be made conformally invariant with an improved energy-momentum tensor which gives a constant energy density. But for the Maxwell field there is no way to construct a local and gauge-invariant analogue of a similar Huggins term to cancel out the position-dependent part of (52). Thus the total Casimir energy obtained by integrating the energy-density is divergent and therefore looks different from what follows from the regularized sum of the zero-point energies of all modes. However, the difference turns out to be an infinite constant independent of the distance between plates.

3.3   Discussion and conclusion

The massless and free, canonical scalar field theory is not conformally invariant in other dimensions than $D = 2$. And it is only then that the Casimir energy density is constant and gives a finite integrated energy. The same satisfactory situation is also possible in higher dimensions when the theory is extended by making it conformally invariant, corresponding to adding the Huggins term to the energy-momentum tensor.

This has been well-known for a long time, but not very well understood from a physical point of view. One of the most recent and detailed discussions of this phenomenon has been undertaken by Fulling who has attempted to understand the divergences in the canonical theory at a deeper level[17]. One can isolate the problem to the lack of commutativity between regularization of the integrated energy and the integration of the regulated energy density. This is perhaps not so surprising from a mathematical point of view, but hard to accept physically since the energy density is a physical quantity and should be tied up with the total energy of system. In other systems like the Casimir energy for a sphere, the energy density again diverges at the surface[18], but this is understood from its non-trivial geometry as first discussed by Deutsch and Candelas[19]. Since then the problem has been addressed by Fulling[17] and Milton with collaborators[20]. For a plane boundary there should be no such geometric complications.

The electromagnetic Casimir effect for $D = 4$ is very similar to the scalar effect for $D = 2$. But for dimensions $D > 4$ there is no Huggins term for the electromagnetic case to cure the problem. From the point of view of the Casimir force alone, this is not a problem because the pressure is given by the expectation value $\langle T_{zz} \rangle$ which is constant in all dimensions and equal to the force. But at first sight this force has little to do with the integral of the energy density $\langle T_{00} \rangle$ which will always diverge at the plates when $D > 4$.

This becomes especially clear when we just consider the electromagnetic fluctuations around one plate. The induced energy density can then be obtained from equation (52)

13

by taking the limit $L \to \infty$. The pressure will then be zero on both sides of the plate while the other components become

$$\langle T_{\bar{\mu}\bar{\nu}} \rangle_{D/N} = \mp (D-2)(D/2-2) \frac{\Gamma(D/2)}{(4\pi)^{D/2}|z|^D} \, \eta_{\bar{\mu}\bar{\nu}} \tag{53}$$

since $f_D = (L/z)^D$ in this limit, as follows from the definition (29). It is non-zero on both sides of the plate and diverges when we approach it. The situation is analogous to the diverging energy density surrounding a classical pointlike electron. Thus, the behaviour (53) is related to the intrinsic structure of a single plate, and the correponding integrated (infinite) energy is part of the energy required to make that plate. It does not contribute to the Casimir force. Thus, to find a connection between Casimir force and energy density it is sufficient to investigate the *changes* in energy density as two plates are brought together from infinite distance. We thus define $\mathcal{T}_{\bar{\mu}\bar{\nu}}$ as the expression (52) subtracted contributions like (53) from plates at $z = (0, L)$, taking into account both sides of each plate. We find that

$$\mathcal{T}_{\bar{\mu}\bar{\nu}}(z) = -(D-2)\frac{\Gamma(D/2)}{(4\pi)^{D/2}L^D} \, \eta_{\bar{\mu}\bar{\nu}} \times \begin{cases} \mp(D/2-2)(L/(L-z))^D & \text{for } z < 0, \\ \zeta_R(D) \pm (D/2-2)\tilde{f}_D(z/L) & \text{for } 0 < z < L, \\ \mp(D/2-2)(L/z)^D & \text{for } z > L, \end{cases} \tag{54}$$

where $\tilde{f}_D(z/L) = \zeta_H(D, 1+z/L) + \zeta_H(D, 2-z/L)$. This quantity $\mathcal{T}_{\bar{\mu}\bar{\nu}}$ is finite everywhere, and its integrated energy agrees perfectly with the Casimir force since the integrals over the z-dependent terms in (54) cancel each other. The consistency between the various approches to the Casimir effect, i.e. total energy from mode sum (2), the pressure term (51) and the change in energy density (54), gives support to the belief in a Casimir force which is essentially independent of the details of the plates.

Of course, the interesting problem of the intrinsic and finite structure of a single plate remains. But this is similar to the problem of resolving the divergences caused by pointlike objects in quantum field theory. One obvious approach would be to modify the boundary conditions. If they were made softer so that they didn't affect fluctuations with wavelengths below a certain cut-off λ_c, one would expect that the resulting energy density would be modified and finite for distances $|z| < \lambda_c$ away from the plate. This has actually been investigated by Graham *et al.* where a more physical boundary is described by an additional field[21].

In the more unphysical case where the Neumann or Dirichlet boundary conditions are replaced by periodic boundary conditions when $D > 4$, there would be no problems of these kinds. The energy density is then constant, giving a total energy consistent with the force between the plates. This is equivalent to the problem of photons in thermal equilibrium. Even if the trace of the energy-momentum tensor is non-zero when $D > 4$, the pressure P in this blackbody radiation is given by the energy density ρ by the standard expression $P = \rho/d$ where $d = D - 1$ is the number of spatial dimensions[22].

But imposing such periodic boundary conditions, would be equivalent to just avoiding the problem. In conclusion, we must admit that the total vacuum fluctuations near confining

14

boundaries is still not completely understood, but is very likely to depend on the microscopic details of those boundaries. This is especially the case for the electromagnetic field in spacetimes with more than four dimensions. Fortunately the Casimir force seems to be rather insensitive to those details.

References

[1] H.B.G. Casimir, *Proc. K. Ned. Akad. Wet.* **51**, 793 (1948).

[2] M.J. Sparnaay, *Physica* **24**, 751 (1958).

[3] S.K. Lamoreaux, *Phys. Rev. Lett.* **78**, 5 (1997); U. Mohideen and A. Roy *ibid.* **81**, 4549 (1998); H.B. Chan *et al. Science* **291**, 1941 (2001); G. Bressi *et al. Phys. Rev. Lett.* **88**, 041804 (2002).

[4] V.V. Nesterenko, G. Lambiase and G. Scarpetta, *Riv. Nuovo Cim.* **27N6**, 1 (2004); I. Brevik, J.B. Aarseth, J.S. Høye and K.A. Milton, *quant-ph/0410231*.

[5] For more general reviews, see for instance M. Bordag, U. Mohideen and V.M. Mostepanenko, *Phys. Rep.* **353**, 1 (2001); K.A. Milton, *J. Phys.* **A37**, R209 (2004).

[6] C.A. Lütken and F. Ravndal, *Phys. Rev.* **A31**, 2082 (1985).

[7] B. DeWitt, *Phys. Rep.* **19C**, 295 (1975).

[8] L.S. Brown and G.J. Maclay, *Phys. Rev.* **184**, 1272 (1969).

[9] A.G. Riess *et al.*, *Astron. J.* **116** 1009 (1998); S. Perlmutter *et al.*, *Astrophys. J.* **517**, 565 (1999).

[10] E. Ponton and E. Poppitz, *JHEP* **0106**, 019 (2001); A. Albrecht, C.P. Burgess, F. Ravndal and C. Skordis, *Phys. Rev.* **D65**, 123507 (2002); K.A. Milton, *Grav. Cosmol.* **9**, 66 (2003).

[11] J. Ambjørn and S. Wolfram, *Ann. Phys.* N.Y. **147**, 1 (1983).

[12] K. Tywoniuk and F. Ravndal, *quant-ph/0408163*.

[13] C.G. Callan, S. Coleman and R. Jackiw, *Ann. Phys. N.Y.* **59**, 42 (1970).

[14] E. Huggins, *Ph.D. thesis*, Caltech, 1962 (unpublished).

[15] K.A. Milton, *Phys. Rev.* **D68**, 065020 (2003).

[16] A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn and V.F. Weisskopf, *Phys. Rev.* **D9**, 3471 (1974).

[17] S.A. Fulling, *J. Phys.* **A36**, 6857 (2003)

[18] K. Olaussen and F. Ravndal, *Nucl. Phys.* **B192**, 237 (1981).

[19] D. Deutsch and P. Candelas, *Phys. Rev.* **D20**, 3063 (1979).

[20] K.A. Milton, I. Cavero-Pelaèz and J. Wagner, *J. Phys.* **A39**, 6543 (2006).

[21] N. Graham, R.L. Jaffe, V. Khemani, M. Quandt, O. Schröder and H. Weigel, *Nucl. Phys.* **B677**, 379 (2004).

[22] H. Alnes, I.K. Wehus and F. Ravndal, *quant-ph/0506131*.